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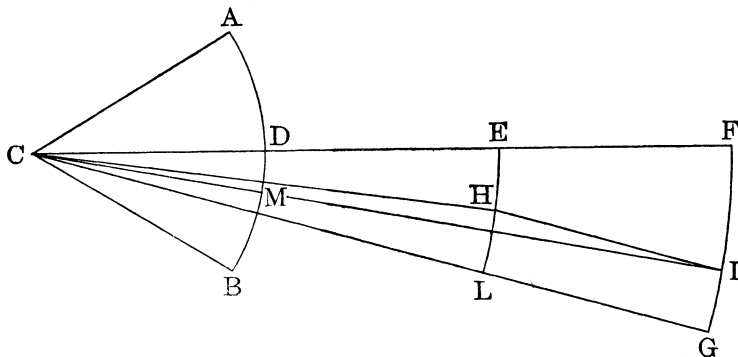
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THE APPROXIMATE INSCRIPTION OF CERTAIN REGULAR POLYGONS.

By PROF. H. A. HOWE, Denver, Col.

1. *To inscribe a regular nonagon in a circle.*—Let ACB be a central angle of 60° in the given circle. Bisect ACB by CF , laying off DE and EF equal to CD .



Bisect FCB by CG , and draw the arcs EL and FG from C as a centre. Bisect FCG , thus determining H , the mid-point of EL . From H draw HI parallel to CG , and from the point I where it cuts FG draw IC . Then will ICG be nearly equal to one third of FCG , or 5° . By solving the triangle HIC we find that $CIH = ICG = 4^\circ 59' 31''.37$. Hence $ACM = 40^\circ 0' 28''.63$, and a chord drawn from A to M will be very nearly equal to the side of a regular nonagon inscribed in the circle whose radius is $AC : 9 \times 28''.63 = 257''.67$, which is $\frac{1}{5030}$ of the entire circumference. If one half the angle FCI were subtracted from an angle of 45° , an angle of $39^\circ 59' 45''.68$ would result, which is a closer approximation to 40° .

It is evident that there are many ways of getting angles more nearly equal to 5° ; we give a few, each of which depends upon the trisection of a small angle, together with successive bisections of angles; thus:

$$5^\circ = \frac{15^\circ}{2} - \frac{1}{3} \cdot \frac{15^\circ}{2} \quad (A)$$

$$= \frac{15^\circ}{4} + \frac{1}{3} \cdot \frac{15^\circ}{4} \quad (B)$$

$$= \frac{15^\circ}{4} + \frac{15^\circ}{8} - \frac{1}{3} \cdot \frac{15^\circ}{8} \quad (C)$$

$$= \frac{15^\circ}{4} + \frac{15^\circ}{16} + \frac{1}{3} \cdot \frac{15^\circ}{16} \quad (D)$$

$$= \frac{15^\circ}{4} + \frac{15^\circ}{16} + \frac{15^\circ}{32} - \frac{1}{3} \cdot \frac{15^\circ}{32} \quad (E)$$

$$= \frac{15^\circ}{4} + \frac{15^\circ}{16} + \frac{15^\circ}{64} + \frac{1}{3} \cdot \frac{15^\circ}{64}, \quad (F)$$

.

If the trisections included in solutions (A), ... (F) be performed in the same manner in which FCG was trisected (getting ICG as the approximate value of $\frac{1}{3}FCG$), it is easy to compute the error of each solution in the following way:—

Let FCG now denote any one of these angles to be trisected; denote ICG by x and HCG by y ; let $2F(y)$ denote $y - \sin y$, and $2F(x)$ denote $x - \sin x$.

Since $\sin y = \frac{3}{2} \sin x$, we have

$$y - 2F(y) = \frac{3}{2}x - \frac{3}{2}.2F(x). \quad (1)$$

By development,

$$2F(y) = \frac{1}{6} \sin^3 y + \frac{3}{40} \sin^5 y + \frac{5}{112} \sin^7 y + \frac{35}{1152} \sin^9 y + \dots \quad (2)$$

Likewise, substituting $\frac{2}{3} \sin y$ for $\sin x$, we get

$$2F(x) = \frac{4}{81} \sin^3 y + \frac{4}{405} \sin^5 y + \frac{40}{15309} \sin^7 y + \frac{140}{177147} \sin^9 y + \dots \quad (3)$$

Hence

$$\frac{2F(x)}{2F(y)} = \frac{8}{27} - (\frac{2}{27} \sin^2 y + \frac{1549}{51030} \sin^4 y + \frac{651827}{41334300} \sin^6 y + \dots) = \frac{8}{27} - \varphi(y), \quad (4)$$

where $\varphi(y)$ denotes the series in the ().

Find the value of $F(x)$ from (4), substitute it in (1), and reduce, obtaining

$$y - \frac{3}{2}x = \frac{5}{9}.2F(y) + \frac{3}{2}\varphi(y).2F(y),$$

or

$$\frac{2}{3}y - x = \frac{10}{27}.2F(y) + \varphi(y).2F(y). \quad (5)$$

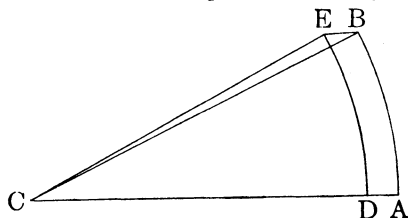
Now $\frac{2}{3}y - x$ is the error of x , and (5) furnishes an easy method of computing this error. Thus we find the following table:—

Solution.	$2y$	Error of x .	$\frac{9 \times \text{error of } x}{1,296,000''}$
	°	''	
(A)	7.5	3.573	1 : 40,290 ±
(B)	3.75	0.4462	1 : 322,700 ±
(C)	1.875	0.0558	1 : 2,583,000 ±
(D)	0.9375	0.00698	1 : 20,660,000 ±
(E)	0.46875	0.00087	1 : 165,000,000 ±
(F)	0.234375	0.00011	1 : 1,300,000,000 ±

It thus becomes apparent that it is possible to find a third of any acute angle without much labor, and attain a result the theoretical error of which is far less than the unavoidable errors of the most careful draughtsmen with the best drawing instruments manufactured. When the angle is between 45° and

90° , it is best to find one third of its complement, and subtract it from 30° . If the given angle is obtuse it is best to find a third of its supplement, and subtract it from 60° .

2. *The Inscription of a Regular Polygon of Eleven Sides.—1st Method.*—

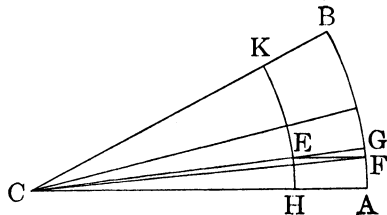


Divide the radius CA of the given circle into twelve equal parts, of which CD contains eleven. Construct $ACB = 30^\circ$. Draw AB and DE . From B draw BE parallel to AC . Draw CE . Then $ECD = 33^\circ 3' 20''$, which is one per cent. larger than $360^\circ : 11$. If the arc ED had been drawn with a radius of 11.1, ECD would have been equal to $32^\circ 43' 13''.60$, which is $\frac{1}{4798}$ smaller than $360^\circ : 11$.

2d Method.—If a triangle be formed of sides 6, 10, 11, the angle opposite 6 will be $32^\circ 45' 50''$, which is $\frac{1}{887}$ larger than $360^\circ : 11$.

3. *The Inscription of a Regular Polygon of Thirteen Sides.*—In the preceding figure let $CD = 12$, and $CA = 13$, and the angle $ECD = 30^\circ$, EB being parallel to AC . Then $ACB = 27^\circ 29' 11''.14$, which is $\frac{10}{1345}$ smaller than $360^\circ : 13$. If CA be taken as $12\frac{10}{11}$, ACB will be $15''.27$ larger than desired, or $\frac{1}{8529}$ too large.

4. *The Multisection of Small Angles.*—If we wish to find one fifth of ACB , by successive bisections we obtain $ACG = \frac{1}{4}BCA$. Lay off $CH = 4$ and $CA = 5$, and draw the arcs BA and HK . From E , where CG cuts HK , draw EF parallel to CA . FCA is nearly one fifth of BCA . Denote GCA by a , and FCA by x . By reasoning similar to that employed before we may show that



$$\frac{4}{5}a - x = \left(\frac{36}{125} + \frac{1296}{15625} \sin^2 a + \dots \right) 2F(a).$$

If $BCA = 15^\circ$, FCA is $2^\circ 59' 57''.22$, which is $2''.78$ too small.

To find $\frac{1}{7}$ of BCA . By successive bisections, find $FCA = \frac{1}{8}BCA$. Lay off $CA = 8$ and $CH = 7$, and draw BA and KH ; also draw FE parallel to CA . ECA is nearly equal to $\frac{1}{7}BCA$. Denoting FCA by a , and ECA by x , we find

$$\frac{8}{7}a - x = \left(-\frac{120}{343} - \frac{3456}{16807} \sin^2 a - \dots \right) 2F(a).$$

If $BCA = 15^\circ$, $ECA = 2^\circ 8' 34''.71$, which is $0''.42$ too large.

By combinations of the methods which have been set forth one may easily find $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{11}$, etc. of any small angle with considerable accuracy. These methods are so simple that it is quite probable that they are not new; but I have never heard of them.